VARIANCE OF PERIODIC MEASURE OF BOUNDED SET WITH RANDOM POSITION

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Abstract. The principal term in the asymptotic expansion of the variance of the periodic measure of ball in $\mathbb{R}^d$ under uniform random shift is proportional to the $(d+1)$st power of the grid scaling factor. This result remains valid for a bounded set in $\mathbb{R}^d$ with sufficiently smooth isotropic covariogram under an uniform random shift and an isotropic rotation, and the asymptotic term is proportional also to a $(d-1)$-dimensional measure of the object boundary. The related coefficients are calculated for various periodic grids constructed from affine sets.

1. Introduction

The area of a planar figure can be estimated by superposing randomly rotated and shifted grid of regularly spaced dots on the image, counting the dots inside the figure and multiplying the number of dots by the grid point specific area. The number of object intersecting grid points is an example of a 2-periodic measure in $\mathbb{R}^2$. Similarly the volume of bounded objects in Euclidean space of an arbitrary dimension can be estimated using any d-periodic measure. The situation can be reversed, namely the grid is fixed and the object moves. The variance of measure of a bounded object shifted and rotated at random can be used to calculate the estimator variance.

The variance of the d-periodic measure of random ball will be calculated and it will be proved, that the conclusion concerning the asymptotic behaviour of the variance of the periodic measure remains valid also for bounded sets with sufficiently smooth isotropic covariograms. The principal term in its asymptotic expansion is proportional to the surface measure of the set with a coefficient depending on the grid. The coefficients of various grids of points, lines or hypersurfaces can be calculated using multidimensional zeta functions.

2. Definitions and Results on Ball

Definition 2.1. Let $\mathbf{T}$ be a discrete subgroup of translations in the d-dimensional Euclidean space $\mathbb{R}^d$. $\mathbf{T}$ can be defined by the regular matrix $A \in \mathbb{R}^{d\times d}$ as $\mathbf{T}(A) = A\mathbb{Z}^d$, where $\mathbb{Z}^d$ is set of all points in $\mathbb{R}^d$ with integral co-ordinates. $\mathbf{T}$ has the fundamental region $F_{\mathbf{T}} = A[0,1)^d$ of volume $\lambda^d(F_{\mathbf{T}}) = \det A$, where λ^d is the Lebesgue measure; hence the spatial intensity of $\mathbf{T}$ is $\alpha = (\det A)^{-1}$.

The group dual to the group $\mathbf{T}(A)$ is $\mathbf{T}^* = \mathbf{T}\left(A^{-1}\right)$.

A $\mathbf{T}$-periodic measure μ in $\mathbb{R}^d$ is a non-negative Borel σ-finite measure such that $\mu(K+x)$ is a $\mathbf{T}$-periodic function of x for any measurable set $K \subseteq \mathbb{R}^d$. The intensity of μ is $\lambda = \alpha\mu(F_{\mathbf{T}})$.

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The Fourier coefficient of a $\mathbf{T}$-periodic measure μ with index $\xi \in \mathbf{T}^*$ is

$$\widetilde{\mu}_\xi = \alpha \int_{F_\mathbf{T}} \exp\left(-2\pi i x \xi\right) d\mu\left(x\right), \tag{2.1}$$

where α is the intensity of $\mathbf{T}$.

Fourier transform of a function $f \in \mathbf{L}^1\left(\mathbb{R}^d\right)$ is

$$\widehat{f}\left(\xi\right) = \int_{\mathbb{R}^d} f\left(x\right) \exp\left(-2\pi i x \xi\right) d\lambda^d\left(x\right). \tag{2.2}$$

If f is moreover spherically symmetric then $r^{d-1} f\left(r\right) \in \mathbf{L}^1\left(\mathbb{R}^+\right)$ and Fourier transform of f can be expressed as the Haenkel transform

$$\widehat{f}\left(\rho\right) = 2\pi \rho^{1-\frac{d}{2}} \int_0^\infty r^{\frac{d}{2}} J_{\frac{d}{2}-1}\left(2\pi \rho r\right) f\left(r\right) dr, \tag{2.3}$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of the first kind.

Notation 2.2. The symbols $\mathbf{E}$ and $\mathbf{Var}$ denote the expected value and the variance, respectively. The convolution of a σ-finite Borel measure μ on $\mathbb{R}^d$ with a function $f \in \mathbf{L}^1\left(\mathbb{R}^d\right)$ with a bounded support is

$$f \star \mu\left(x\right) = \int_{\mathbb{R}^d} f\left(x-y\right) d\mu\left(y\right). \tag{2.4}$$

Theorem 2.3. *Let μ be a $\mathbf{T}$-periodic measure and let K be a bounded measurable set in $\mathbb{R}^d$. Then*

$$\mathbf{E}\left(I_K \star \mu\right) \equiv \int_{F_\mathbf{T}} \left(I_K \star \mu\right) \alpha d\lambda^d = \lambda \lambda^d\left(K\right), \tag{2.5}$$

and

$$\mathbf{Var}\left(I_K \star \mu\right) \equiv \int_{F_\mathbf{T}} \left(I_K \star \mu - \mathbf{E}\left(I_K \star \mu\right)\right)^2 \alpha d\lambda^d = \sum_{\xi \in \mathbf{T}^*}^{\xi \neq 0} \left|\widetilde{\mu}_\xi\right|^2 \left|\widehat{I_K}\left(\xi\right)\right|^2, \tag{2.6}$$

where α is the spatial density of $\mathbf{T}$ and λ is the intensity of μ.

Proof. Equality (2.5) can be proved by standard aguments. We have from (2.4) and periodicity of μ

$$\int_{F_\mathbf{T}} \int_{\mathbb{R}^d} I_K\left(x-y\right) d\mu\left(y\right) \alpha d\lambda^d\left(x\right) = \int_{F_\mathbf{T}} \int_{F_\mathbf{T}} \sum_{z \in \mathbf{T}} I_{K+z}\left(x-y\right) d\mu\left(y\right) \alpha d\lambda^d\left(x\right).$$

By changing the integration order using Fubini theorem we get

$$\alpha \int_{F_\mathbf{T}} \int_{F_\mathbf{T}} \sum_{z \in \mathbf{T}} I_{K+z}\left(x-y\right) d\lambda^d\left(x\right) d\mu\left(y\right) = \alpha \mu\left(F_\mathbf{T}\right) \int_{\mathbf{R}^d} I_K d\lambda^d = \lambda \lambda^d\left(K\right).$$

Equality (2.6) follows from the Parseval theorem, because $I_K \star \mu \in \mathbf{L}^2\left(F_\mathbf{T}\right)$ and the functions $\exp\left(-2\pi i x \xi\right)$, $\xi \in \mathbf{T}^*$, form an orthonormal base in $\mathbf{L}^2\left(F_\mathbf{T}, \alpha \lambda^d\right)$. □

Definition 2.4. Covariogram of a bounded measurable set K is the function $\gamma_K = I_K \star I_{-K}$. It follows from the properties of Fourier transforms that $\widehat{\gamma_K} = \left|\widehat{I_K}\right|^2$ is a nonnegative function. The isotropic covariogram is $\overline{\gamma_K}\left(|u|\right) = \mathbf{E}_M \gamma_{MK}\left(u\right)$ where MK is the set K rotated by $M \in \mathbf{SO}_d$ and the mean $\mathbf{E}_M$ is calculated by integration using the invariant probabilistic measure on $\mathbf{SO}_d$, the group of rotations in $\mathbb{R}^d$; an equivalent definition is $\overline{\gamma_K}(v) = \mathbf{E}_{u,|u|=v}\gamma_K(u)$. The Haenkel transform of the isotropic covariogram is $\widehat{\overline{\gamma_K}}$.

Remark 2.5. It follows from the definition that γ_K is bounded and, as $\widehat{\gamma_K} \geq 0$, the function $\widehat{\gamma_K}$ is integrable in $\mathbb{R}^d$(see [3] Theorem 9.). Further, $\rho^{d-1}\widehat{\overline{\gamma_K}}(\rho) \geq 0$ is integrable in $\mathbb{R}^+$ by Fubini theorem. γ_K is then the inverse Fourier transform 2.2 of $\widehat{\gamma_K}$ ([3] Theorem 8.) and $\overline{\gamma_K}$ is the (inverse) Haenkel transform 2.3 of $\widehat{\overline{\gamma_K}}(\rho)$. By the variance decomposition Lemma [9] (the variance is the variance of the conditional mean plus the mean of conditional variance) we have from (2.6)

$$\mathbf{E}_{M\in SO_d}\mathbf{Var}\left(I_{MK}\star\mu\right) = \sum_{\xi\in\mathbf{T}^*}^{\xi\neq 0} \left|\widetilde{\mu}_\xi\right|^2 \widehat{\overline{\gamma_K}}\left(|\xi|\right)$$

as the variance of the conditional mean is zero here.

The variance of the estimate of volume of the ball by a periodic measure can be calculated using Bessel functions of the first kind. D. G. Kendall and R. A. Rankin in [5], [6] used this approach to study the variance of the area estimate of ovals in plane and of volume estimate of ball by point grids in an arbitrary dimension. A straightforward generalization of their results to periodic measures is given in what follows.

$$\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \tag{2.7}$$

is the volume of the unit ball $B_d(1)$ in $\mathbb{R}^d$.

Lemma 2.6. *The Fourier transform of the characteristic function of the ball $B_d(R)$ with diameter $R > 0$ in $\mathbb{R}^d$ is*

$$\widehat{I_{B_d(R)}}(\xi) = \left(\frac{R}{|\xi|}\right)^{\frac{d}{2}} J_{\frac{d}{2}}\left(2\pi R|\xi|\right),$$

where J_ν is the Bessel function of the first kind. For $(R|\xi|) \to +\infty$

$$\widehat{I_{B_d(R)}}^2(\xi) = \frac{1}{2\pi^2}\frac{R^{d-1}}{|\xi|^{d+1}}\left(1+\cos\left(4\pi R|\xi| - (d+1)\frac{\pi}{2}\right) + o(1)\right).$$

Proof. The first equation follows from the Poisson integral [13] 3.3(3)

$$\int_{|x|<r} \exp\left(2\pi i x\xi\right) d\lambda^d(x) = \left(\frac{r}{|\xi|}\right)^{\frac{d}{2}} J_{\frac{d}{2}}\left(2\pi r|\xi|\right).$$

The second equation follows from the first one and from the asymptotic expansion of Bessel function of the first kind for $z \to \infty$ [13] 7.21(1): $J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{(2\nu+1)\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right)$. □

Now we can proceed to the asymptotic expansion of the variance of the volume estimator using homothetic image of the periodic measure with scale factor $u \to 0+$. The following notation is introduced to simplify the statements of the related theorems.

Notation 2.7. Let μ be a $\mathbf{T}$-periodic measure in $\mathbb{R}^d$, $u \in \mathbb{R}^+$, $K \subseteq \mathbb{R}^d$ measurable. Then the u-scaled measure $\mu_u(K) = u^d \mu\left(u^{-1}K\right)$ is $u\mathbf{T}$-periodic.

Theorem 2.8. *Let μ be a $\mathbf{T}$-periodic measure, $u \in \mathbb{R}^+$. Then*

$$\mathbf{E}\left(I_{B_d(R)} \star \mu_u\right) = \lambda \kappa_d R^d, \tag{2.8}$$

$$\begin{aligned}\mathbf{Var}\left(I_{B_d(R)} \star \mu_u\right) &= \sum_{\xi \in \mathbf{T}^*}^{\xi \neq 0} \left|\widetilde{\mu}_\xi\right|^2 \left(\frac{R}{u^{-1}|\xi|}\right)^d J_{\frac{d}{2}}^2\left(2\pi R u^{-1}|\xi|\right) = \\ &= \frac{R^{d-1}}{2\pi^2}\left(\sum_{\xi \in \mathbf{T}^*}^{\xi \neq 0} \frac{\left|\widetilde{\mu}_\xi\right|^2}{|\xi|^{d+1}}\right) \Phi\left(Ru^{-1}\right) u^{d+1},\end{aligned} \tag{2.9}$$

where Φ defined by the above equality fulfills
$\lim_{x\to\infty} \frac{1}{x}\int_0^x \Phi(x)\, dx = 1$, $0 \le \Phi$, $\limsup_{x\to\infty} \Phi(x) \le 2$.

Proof. It follows from Theorem 2.3 and Lemma 2.6. See also [6] □

Notation 2.9. Equality (2.9) can be expressed using the surface measure of the ball, $H^{d-1}\left(\partial B_d(R)\right)$, and the constant C_μ^V

$$\begin{aligned}\mathbf{Var}\left(I_{B_d(R)} \star \mu_u\right) &= C_\mu^V H^{d-1}\left(\partial B_d(R)\right) \Phi\left(Ru^{-1}\right) u^{d+1}, \\ C_\mu^V &= \frac{1}{2\pi^2 d\kappa_d} \sum_{\xi \in \mathbf{T}^*}^{\xi \neq 0} \frac{\left|\widetilde{\mu}_\xi\right|^2}{|\xi|^{d+1}}.\end{aligned} \tag{2.10}$$

Matérn studied in [7] numerically the variance of estimate of various figures in plane by grids of points or lines and proposed the validity of the above formula for a large class of figures. Matheron formulated in his transitive theory [8] asymptotic results for orthogonal point grids in an arbitrary dimension and found an approximation of the relevant coefficients. The rest of the article is devoted to the generalization of (2.10) for some other bounded objects and to the calculation of the coefficients C_μ^V for various grids.

3. Asymptotic Expansion of Variance of Periodic Measure of Randomly Placed Bounded Set

Definition 3.1. A function f is in $\mathbf{BV}^s(\mathbb{R}^+)$, $s \ge 0$, iff there is a finite signed measure σ on $\mathbb{R}^+$ such that f is a fractional integral of the Weyl type:

$$f(x) = \frac{1}{\Gamma(s+1)} \int_x^\infty (y-x)^s \, d\sigma(y)$$

for $x \in \mathbb{R}^+$, i.e. iff $f^{(s)}$, the (generalized) derivative of the order s, has a bounded variation. A function is in $\mathbf{BV}_c^s(\mathbb{R}^+)$ iff it is in $\mathbf{BV}^s(\mathbb{R}^+)$ and has a bounded support.

Remark 3.2. a) $s \ge 1$: f is in $\mathbf{BV}^s(\mathbb{R}^+)$ iff f' is in $\mathbf{BV}^{s-1}(\mathbb{R}^+)$. b) The covariogram of the ball is in $\mathbf{BV}_c^{\frac{d+1}{2}}(\mathbb{R}^+)$.

Proof. a) follows from the differentiation of $\frac{1}{\Gamma(s+1)}\int_x^\infty (y-x)^s \, d\sigma(y)$ under the integral. b) $\overline{\gamma_B}'(r) = \kappa_{d-1}\left(1-r^2\right)^{\frac{d-1}{2}} = f(r)(1-r)^{\frac{d-1}{2}}$, where $f = \kappa_{d-1}(1+r)^{\frac{d-1}{2}}$ is smooth in $\mathbb{R}^+$ and $(1-r)^{\frac{d-1}{2}}$ is in $\mathbf{BV}_c^{\frac{d-1}{2}}(\mathbb{R}^+)$. □

Lemma 3.3. *If $\beta > \alpha - \frac{1}{2}$ and $\alpha + \nu > 0$, then for $x \to +\infty$*

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} J_\nu(xt)\, dt = \frac{2^{\alpha-1}\Gamma\left(\frac{1}{2}(\alpha+\nu)\right)}{\Gamma\left(1-\frac{1}{2}(\alpha-\nu)\right)} x^{-\alpha} + o\left(x^{-\alpha}\right).$$

Proof. From

$$x^{\alpha}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}J_{\nu}(xt)\,dt=\int_0^x y^{\alpha-1}\left(1-\frac{y}{x}\right)^{\beta-1}J_{\nu}(y)\,dy$$

by integration per partes using $\int x^{\nu}J_{\nu-1}(x)\,dx=x^{\nu}J_{\nu}(x)$ and taking into account Weber integral [13] 13.24(1)

$$\int_0^{\infty} y^{\mu-1}J_{\nu}(y)\,dy=\frac{2^{\mu-1}\Gamma\left(\frac{1}{2}(\mu+\nu)\right)}{\Gamma\left(1-\frac{1}{2}(\mu-\nu)\right)}$$

with $\mu<\frac{3}{2}$ and $\mu+\nu>0$. See also [11] 10.86 . □

Notation 3.4. $\mathbf{Var}(\mu_u,K)$ is the variance of the periodic measure μ_u of a uniformly randomly shifted and isotropically rotated set K.

Remark 3.5. If K is a bounded full-dimensional locally finite union of sets of finite reach (eg. polyhedron, set with piecewise $\mathbf{C}^2$ smooth boundary or finite union of full-dimensional convex sets), then $-\overline{\gamma_K}'^{+}(0)=\frac{\kappa_{d-1}}{d\kappa_d}H^{d-1}(\partial K)$ [10].

Theorem 3.6. *Let μ be a $\mathbf{T}$-periodic measure, $u\in\mathbb{R}^+$, K a bounded measurable set such that finite $\overline{\gamma_K}'^{+}(0)$ exists and Φ a function on $\mathbb{R}^+$ defined by equation*

$$\mathbf{Var}(\mu_u,K)=\frac{-\overline{\gamma_K}'^{+}(0)}{2\pi^2\kappa_{d-1}}\left(\sum_{\xi\in\mathbf{T}^*}^{\xi\neq0}\frac{|\widetilde{\mu}_{\xi}|^2}{|\xi|^{d+1}}\right)\Phi\left(u^{-1}\right)u^{d+1}.$$

Then

i) if $\overline{\gamma}_K$ is in $\mathbf{BV}_c^{\frac{d+1}{2}}(\mathbb{R}^+)$ then

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x\Phi(x)\;dx=1,\tag{3.1}$$

ii) if $\overline{\gamma}_K$ is in $\mathbf{BV}_c^{\frac{d+3}{2}}(\mathbb{R}^+)$ then

$$\lim_{x\to\infty}\Phi(x)=1.\tag{3.2}$$

Proof. By 2.5 we have

$$\mathbf{Var}(\mu_u,K)=\mathbf{E}_{M\in SO_d}\mathbf{Var}(I_{MK}\star\mu_u)=\sum_{\xi\in\mathbf{T}^*}^{\xi\neq0}|\widetilde{\mu}_{\xi}|^2\,\widehat{\overline{\gamma_K}}\left(u^{-1}|\xi|\right).$$

We shall prove first that the auxiliary function Ψ defined by the equation

$$-\overline{\gamma_K}'^{+}(0)\,\Psi(x)=2\pi^2\kappa_{d-1}x^{d+1}\widehat{\overline{\gamma_K}}(x)$$

has the property (3.1) or (3.2). It is easy to see that the function

$$\Phi(x)=\frac{\sum_{\xi\in\mathbf{T}^*}^{\xi\neq0}c_{\xi}\Psi(|\xi|x)}{\sum_{\xi\in\mathbf{T}^*}^{\xi\neq0}c_{\xi}},\quad c_{\xi}=\frac{|\widetilde{\mu}_{\xi}|^2}{|\xi|^{d+1}},$$

has then the same property too.

ad i) Let $\overline{\gamma}_K$ be in $\mathbf{BV}_c^{\frac{d+1}{2}}(\mathbb{R}^+)$. Then Remark 2.5 and the change of integration order yield

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R 2\pi^2\kappa_{d-1}\rho^{d+1}\widehat{\overline{\gamma_K}}(\rho)\,d\rho=$$

$$=\lim_{R\to\infty}\int_0^{\infty}\overline{\gamma_K}(r)\,\frac{1}{R}\int_0^R 4\pi^3\kappa_{d-1}\rho^{\frac{d}{2}+2}r^{\frac{d}{2}}J_{\frac{d}{2}-1}(2\pi r\rho)\,d\rho dr$$

and the subsequent integration by parts followed by $\int x^{\nu} J_{\nu-1}(x)\,dx = x^{\nu} J_{\nu}(x)$ gives

$$\lim_{R\to\infty} \int_0^\infty -\overline{\gamma_K}'(r)\,\frac{1}{R}\int_0^R 2\pi^2\kappa_{d-1}\rho^{\frac{d}{2}+1} r^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi r\rho)\,d\rho dr =$$

$$= \lim_{R\to\infty}\int_0^\infty -\overline{\gamma_K}'(r)\,\pi\kappa_{d-1}R^{\frac{d}{2}} r^{\frac{d}{2}-1} J_{\frac{d}{2}+1}(2\pi rR)\,dr.$$

From the assumption that $\overline{\gamma}_K$ is in $\mathbf{BV}_c^{\frac{d+1}{2}}(\mathbb{R}^+)$ follows the existence of a signed measure σ with bounded support such that $\overline{\gamma_K}'(r) = \Gamma\left(\frac{d+1}{2}\right)^{-1}\int_r^\infty (t-r)^{\frac{d-1}{2}}\,d\sigma(t)$ and by changing the integration order we obtain

$$-\pi\kappa_{d-1}\Gamma\left(\frac{d+1}{2}\right)^{-1}\lim_{R\to\infty} R^{\frac{d}{2}}\int_0^\infty\int_0^t (t-r)^{\frac{d-1}{2}}\, r^{\frac{d}{2}-1} J_{\frac{d}{2}+1}(2\pi rR)\,dr d\sigma(t).$$

Finally, the substitution $r = ty$ and Lemma 3.3 give

$$-\Gamma\left(\frac{d+1}{2}\right)^{-1}\int_0^\infty t^{\frac{d-1}{2}}\,d\sigma(t) = -\overline{\gamma_K}'^{+}(0).$$

ad ii) Let $\overline{\gamma}_K$ be in $\mathbf{BV}_c^{\frac{d+3}{2}}(\mathbb{R}^+)$. Remark 2.5 and the change of the integration order yield

$$\lim_{R\to\infty} 2\pi^2\kappa_{d-1}R^{d+1}\widehat{\overline{\gamma_K}}(R) =$$

$$= \lim_{R\to\infty}\int_0^\infty \overline{\gamma_K}(r)\,4\pi^3\kappa_{d-1}R^{\frac{d}{2}+2} r^{\frac{d}{2}} J_{\frac{d}{2}-1}(2\pi rR)\,dr.$$

By integration by parts and using $\int x^{\nu} J_{\nu-1}(x)\,dx = x^{\nu} J_{\nu}(x)$ we get

$$\lim_{R\to\infty}\int_0^\infty -\overline{\gamma_K}'(r)\,2\pi^2\kappa_{d-1}R^{\frac{d}{2}+1} r^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi rR)\,dr.$$

From the assumption that $\overline{\gamma}_K$ is in $\mathbf{BV}_c^{\frac{d+3}{2}}(\mathbb{R}^+)$ follows the existence of a signed measure σ with bounded support such that $\overline{\gamma_K}'(r) = \Gamma\left(\frac{d+3}{2}\right)^{-1}\int_r^\infty (t-r)^{\frac{d+1}{2}}\,d\sigma(t)$ and by changing the integration order we obtain

$$-2\pi^2\kappa_{d-1}\Gamma\left(\frac{d+3}{2}\right)^{-1}\lim_{R\to\infty} R^{\frac{d}{2}+1}\int_0^\infty\int_0^t (t-r)^{\frac{d+1}{2}}\, r^{\frac{d}{2}} J_{\frac{d}{2}}(2\pi rR)\,dr d\sigma(t).$$

Finally, by substitution $r = ty$ and using Lemma 3.3 we get

$$= -\Gamma\left(\frac{d+3}{2}\right)^{-1}\int_0^\infty t^{\frac{d+1}{2}}\,d\sigma(t) = -\overline{\gamma_K}'^{+}(0).$$

□

Corollary 3.7. *From Theorem 3.6 and Remark 3.5 it follows that*

$$\mathbf{Var}(\mu_u, K) = C_\mu^V H^{d-1}(\partial K)\,\Phi\left(u^{-1}\right) u^{d+1}$$

with coefficients C_μ^V defined in (2.10) and Φ fulfills either (3.1) or (3.2) according to the regularity of isotropic covariogram of K.

4. Evaluation of Coefficients of Grids of Affine Sets

If μ is the counting measure on $d-$periodic grid of points $A\mathbb{Z}^d$ the coefficients C_μ^V introduced in Notation 2.9 and Corollary 3.7,

$$C_\mu^V = \frac{1}{2\pi^2 d\kappa_d} \sum_{n\in\mathbb{Z}^{\mathbf{d}}}^{n\neq 0} \left|A^{-1}n\right|^{-d-1},$$

can be calculated using Epstein zeta function

$$Z\left(A^{-1},s\right) = \sum_{n\in\mathbb{Z}^{\mathbf{d}}}^{n\neq 0} \left|A^{-1}n\right|^{-s}.$$

Only grids of intensity $\alpha = 1$ will be studied as it makes possible a straightforward comparison of the efficiency of the related volume estimators and the results for general grids can be obtained by scaling.

For hypercubic grids of points in $\mathbb{R}^d$ $A = I_d$, where I_d is the identity matrix. For (self-dual) triangular grid of points in $\mathbb{R}^2$

$$A = A_2 = \frac{\sqrt{2}}{\sqrt[4]{3}} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Face centered cubic grid and body centered cubic grid of points in $\mathbb{R}^3$ are mutually dual with matrices $A = D_3$ and $A = D_3^*$, respectively:

$$D_3 = \frac{1}{\sqrt[3]{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},\ D_3^* = \frac{1}{\sqrt[3]{4}} \begin{pmatrix} -1 & +1 & +1 \\ +1 & -1 & +1 \\ +1 & +1 & -1 \end{pmatrix}.$$

The lattices of the closest packings of spheres in dimensions $d = 4, 8, 24$ are D_4, E_8, Λ_{24} [12].

Using the Mellin transform and the Poisson summation

$$\sum_{n\in\mathbb{Z}^{\mathbf{d}}} \exp -\pi \left|A^{-1}n\right|^2 t = \det A t^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^{\mathbf{d}}} \exp -\pi \left|An\right|^2 t^{-1}$$

we obtain the Riemann expansion

(4.1)

$$\Gamma\left(\tfrac{s}{2}\right)\pi^{-\frac{s}{2}} Z\left(A^{-1},s\right) = \tfrac{2\det A}{s-d} - \tfrac{2}{s} + \sum_{n\in\mathbb{Z}^{\mathbf{d}}}^{n\neq 0} \Gamma\left(\tfrac{s}{2}, \pi\left|A^{-1}n\right|^2\right)\left(\pi\left|A^{-1}n\right|^2\right)^{-\frac{s}{2}} + $$
$$+\det A \sum_{n\in\mathbb{Z}^{\mathbf{d}}}^{n\neq 0} \Gamma\left(\tfrac{d-s}{2}, \pi\left|An\right|^2\right)\left(\pi\left|An\right|^2\right)^{\frac{s-d}{2}},$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is an incomplete gamma function. The function $Z\left(A^{-1},s\right)$ can be evaluated with the precision of the order of $e^{-\pi L^2}$ by summing all terms with $|An| < L$, $\left|A^{-1}n\right| < L$ [3].

Various identities valid between special Epstein zeta functions, the Riemann zeta function ζ and Dirichlet function L_p

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad L_p(s) = \sum_{n=0}^{\infty} (p|n)\, n^{-s}$$

(where $(p|n)$ is Kronecker symbol from number theory) can also be used for calculation of the Epstein zeta functions:

$Z(I_2,s) = 4\zeta\left(\frac{s}{2}\right)L_{-4}\left(\frac{s}{2}\right)$ [5],

$Z(A_2,s) = 2^{1+\frac{s}{2}}3^{1-\frac{s}{4}}\zeta\left(\frac{s}{2}\right)L_{-3}\left(\frac{s}{2}\right)$ [6],
$Z(I_4,s) = 8\left(1-2^{2-s}\right)\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-1\right)$ [2],
$Z(D_4,s) = 24\left(1-2^{1-\frac{s}{2}}\right)2^{-\frac{s}{4}}\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-1\right)$ from theta function in [12],
$Z(I_6,s) = 16\zeta\left(\frac{s}{2}\right)L_{-4}\left(\frac{s}{2}-2\right) - 4\zeta\left(\frac{s}{2}-2\right)L_{-4}\left(\frac{s}{2}\right)$ [2],
$Z(I_8,s) = 16\left(1-2^{1-\frac{s}{2}}+2^{4-s}\right)\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-3\right)$ [2],
$Z(E_8,s) = 240\cdot 2^{-\frac{s}{2}}\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-3\right)$ from theta function in [12],
$Z(I_{24},s) = \frac{16}{691}\left(1-2^{1-\frac{s}{2}}+2^{12-s}\right)\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-11\right)+$
$+\frac{128}{691}\left(259+745\cdot 2^{4-\frac{s}{2}}+259\cdot 2^{12-s}\right)g_{24}\left(\frac{s}{2}\right)$ [2],
$Z(\Lambda_{24},s) = \frac{65520}{691}\left(\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-11\right)-g_{24}\left(\frac{s}{2}\right)\right)$ from theta function in [12], where $g_{24}(t) = \sum_{n=1}^{\infty}\tau(n)\,n^{-t}$ is Ramanujan - Dirichlet function and $\sum_{i=0}^{\infty}\tau(n)\,q^n = q\prod_{i=1}^{\infty}\left(1-q^i\right)^{24}$. Unfortunately, no similar relation is known for any three-dimensional grid.

Grids of parallel affine sets of the dimension k can be calculated using the zeta functions of point grids in the dimension $d-k$.

The coefficients of grids of parallel lines in $\mathbb{R}^3$ intersecting a perpendicular plane in a square or a triangular grid of points are calculated from the zeta functions $Z(I_2,4)$ or $Z(A_2,4)$, respectively.

For grids of parallel hyper-surfaces in $\mathbb{R}^d$ the $Z(I_1,d+1) = 2\zeta(d+1)$ where $\zeta(s)$ is the Riemann zeta function.

Fourier coefficients of shifted grids and combinations of grids can be obtained by linear operations with the Fourier coefficients of the grids.

Let the multiple grids of lines in $\mathbb{R}^3$ be expressed parametrically as $T_i = \mathbf{o}_i + f\mathbf{v}_i + g\mathbf{h}_i + \alpha\mathbf{d}_i$, $i = 0\ldots n$, f and g are integers, α is real and $\mathbf{o}_i$, $\mathbf{v}_i$, $\mathbf{h}_i$, $\mathbf{d}_i$ are vectors from $\mathbb{R}^3$.

The grid of unit density with square cross-section in $\mathbb{R}^3$ [4] is composed of three orthogonal sets of parallel lines, $\mathbf{d}_i = \mathbf{e}_i$, $i = 1,2,3$, $\mathbf{v}_1 = \sqrt{3}\mathbf{e}_3$, $\mathbf{v}_2 = \mathbf{v}_3 = \sqrt{3}\mathbf{e}_1$, $\mathbf{h}_1 = \mathbf{h}_3 = \sqrt{3}\mathbf{e}_2$, $\mathbf{h}_2 = \sqrt{3}\mathbf{e}_3$, the sum in 2.10 is $3Z(I_2,4)+12\zeta(4)$ for self-intersecting grid $\mathbf{o}_i = 0$, $i = 1..3$ and $3Z(I_2,4)-\frac{21}{2}\zeta(4)$ for grid optimized by mutually shifting the collections $\mathbf{o}_1 = 0$, $\mathbf{o}_2 = \frac{\sqrt{3}}{2}\mathbf{e}_3$, $\mathbf{o}_3 = \frac{\sqrt{3}}{2}(\mathbf{e}_1+\mathbf{e}_2)$.

For a quadruple of sets of parallel lines with triangular cross-section and directions of diagonals of the cube $\mathbf{d}_1 = \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$, $\mathbf{d}_2 = \mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3$, $\mathbf{d}_3 = \mathbf{e}_1-\mathbf{e}_2+\mathbf{e}_3$, $\mathbf{d}_4 = \mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3$, $\mathbf{v}_i = 2^4\sqrt{3}\mathbf{e}_2$, $\mathbf{h}_i = 2^4\sqrt{3}\mathbf{e}_3$, $i = 1,2,3,4$, the sum in 2.10 is $4Z(A_2,4)+18\zeta(4)$ for self-intersecting grid $\mathbf{o}_i = 0$, $i = 1,2,3,4$, and $4Z(A_2,4)-\frac{63}{4}\zeta(4)$ for optimized grid $\mathbf{o}_1 = \sqrt[4]{3}(\mathbf{e}_1+\mathbf{e}_2)$, $\mathbf{o}_2 = \sqrt[4]{3}(\mathbf{e}_1+\mathbf{e}_3)$, $\mathbf{o}_3 = 0$, $\mathbf{o}_4 = \sqrt[4]{3}(\mathbf{e}_2+\mathbf{e}_3)$.

The values of the constant C_μ^V for various grids with unit spatial density of the corresponding Hausdorff measure are in tables 1 and 2, where d is the dimension of embedding space and k is the dimension of the affine sets. The values of $Z\left(A^{-1},s\right)$ were calculated by 4.1 and from the above identities for Zeta functions. The procedure 4.1 could be applied for lattices up to I_8, the values of $Z(E_8,9)$, $Z(I_{24},25)$, $Z(\Lambda_{24},25)$ were evaluated from the identities only. The triangular grid and the body centered cubic grid have the smallest observed coefficients of grids of points in $d = 2,3$ and are the duals to the grids of the closest sphere packings. As such a relation may be more general, the coefficients of duals of the closest sphere packings in $d = 4,8,24$ were also evaluated.

d	k	Grid	C_μ^V
1	0	I_1	0.083333333
2	0	square I_2	0.072837040
2	0	triangular A_2	0.071701169
2	1	parallel lines I_2	0.019384090
3	0	cubic I_3	0.066649070
3	0	body centered cubic D_3^*	0.064350404
3	0	face centered cubic D_3	0.064389706
3	1	parallel lines, I_2 cross-section	0.024296742
3	1	parallel lines, A_2 cross-section	0.023315276
3	1	lines, I_2 cross-section, triple	0.125250104
3	1	lines, I_2 cross-section, optimal triple	0.027075333
3	1	lines, A_2 cross-section, quadruple	0.171800922
3	1	lines, A_2 cross-section, optimal quadruple	0.024538766
3	2	parallel planes	0.008726646

Table 1. Coefficients of grids of affine sets of dimension k in $\mathbb{R}^d$, $d \leq 3$.

d	Grid	C_μ^V
4	I_4	0.062959415
4	D_4	0.058670401
5	I_5	0.061045829
6	I_6	0.060656899
7	I_7	0.061828449
8	I_8	0.064852630
8	E_8	0.045596961
24	I_{24}	52.76720063
24	Λ_{24}	0.028950578

Table 2. Coefficients of grids of points in $\mathbb{R}^d$.

5. Conclusions

The asymptotic expansion of the variance of the estimators of volume of bounded objects (Corollary 3.7) have been used for a long time in stereological studies. Supposing some smoothness of the covariograms of the objects, the expansion follows from integral geometric identities. Such smoothness is proved for balls and can be conjectured for bounded objects with smooth boundary. The coefficients of periodic grids of affine sets can be calculated using multidimensional zeta function.

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