ERRORS OF SPATIAL GRIDS ESTIMATORS OF VOLUME AND SURFACE AREA

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ABSTRACT

Errors of the volume and surface area estimators that use IUR spatial grids of points, lines and planes are calculated using geometric characteristics of the objects under study. Coefficients in trend terms of asymptotic formulae for variances of volume estimators that use simple orthogonal grids are calculated. The variance of surface area estimation using spatial grids of lines has two components, the variance due to orientation and the residual variance. Upper bounds of the component of the error of surface area estimator due to the orientation are calculated for various sets of directions of grid lines. Coefficients in trend terms of asymptotic formulae for residual variances of the surface area estimators using triple orthogonal spatial grids of lines are calculated.

Key words: spatial grid, systematic sampling, surface area, variance, volume

INTRODUCTION

Isotropic uniform random (IUR) periodic spatial grids consisting of points, lines or planes (Fig. 1) can be used for estimation of volume, surface area or length of solid objects (Cruz-Orive, 1997). Formulae for prediction of error of the estimators from the grid density and known geometrical properties of the object under study are useful for the design of efficient measurements. An example is the formula for prediction of the error of point counting method of area estimation in plane from the perimeter of the feature set (Gundersen and Jensen, 1987).

Fig. 1. Simple spatial grids of points, lines and planes.

The first part of the paper deals with the volume estimation by measurement of the intersection of the grid and the object. Formulae for the variance of the sphere volume estimation by a spatial grid of points (Kendall and Rankin, 1953) can be easily generalised to IUR oriented grids of lines and planes. The variance as the function of the grid spacing is the sum of the trend term and of the oscillating term (Matheron, 1965). The formula for the trend term can be used for arbitrary objects with known surface area.

The surface area can be estimated from the number of intersections of an IUR grid and the surface (Sandau, 1987). The variance of a surface area estimator using an IUR spatial grid of lines can be decomposed to the component due to orientation (o) of the grid and to the residual component (Hahn and Sandau, 1989).

$$\operatorname{var} \mathrm{est} = \underset{o}{\operatorname{var}}\left(\mathrm{E}(\mathrm{est} \mid o)\right) + \underset{o}{\mathrm{E}}\left(\operatorname{var}(\mathrm{est} \mid o)\right)$$

Counting the intersections of the surface and the spatial grid of lines is equivalent to projecting the surface to planes perpendicular to directions of the grid lines and then counting points of a planar grid inside the total projection. The component of variance due to the orientation of the grid is equal to the variance of the surface area estimator from the total projections based on the Cauchy formula and it depends on the anisotropy of the surface, on the rose of directions of normals to the surface (Appendix B). Using grids consisting of lines in more directions can decrease this component of variance. Values of coefficient of error for flat objects, representing upper bounds for all objects, are calculated for various systematic sets of directions in the second part of the paper. Results can be used for a comparison of efficiency of measurements using either spatial grids of lines with the sets of directions or of measurements using projections of surface into these directions.

The trend term of asymptotic approximation of the residual variance can be calculated for spheres and the result can be generalised to convex objects. Increasing the grid density decreases the residual variance. Calculation of coefficients in the trend term of estimators using triple orthogonal grids of lines is the aim of the third part of the paper.

VOLUME ESTIMATION USING IUR SIMPLE SPATIAL GRIDS

Volume of the object X can be estimated from k-dimensional measure of the part of a simple k-dimensional grid G (Fig. 1) contained inside the object:

$$\mathrm{estV}(X) = \nu_k (X \cap G) u^{d-k}$$

where d is dimension of the reference space, k is dimension of the grid and u the grid spacing. The formula for the trend term in asymptotic expansion of variance of volume estimator of the sphere B_R in R_d using the simple k-dimensional grid (Appendix A):

$$\operatorname{var}(\mathrm{estV}(B_R)) \cong C_{d,k} S(\delta B_R) u^{d+1} \tag{1}$$

is valid for arbitrary objects with finite surface area, because the trend term depends only on the value of the derivative of the geometric covariogram in 0 (Matheron, 1971). Values of some of the coefficients $C_{d,k}$ calculated according to Appendix A (Tab.1) can be compared to values reported by other authors. C_{10}, C_{21} and C_{32} are well known from transitive theory (Matheron, 1965). C_{20} and C_{21} can be found in Matern (1989). The value of C_{30} is slightly higher than that obtained by approximation after Matheron (1965, Cruz-Orive 1989) equal to 0.0656 and is close to numerical estimates for ellipsoids 0.06646 - 0.06684 calculated by Kellerer (Cruz-Orive 1989). The value of C_{40} : 0.06296 is also closer to the numerical estimate 0.0632 by Matern (1989) than approximation after Matheron (1965, Cruz-Orive 1989) equal to 0.0610.

Tab. 1. Values of coefficients $C_{d,k}$, for estimation of d-dimensional volume in d-dimensional reference space by the grid of dimension k (Formula 1).

R_d / k space / grid of:	0 points	1 lines	2 planes
line (R_1)	0.08333	-	-
plane (R_2)	0.07284	0.01938	-
space (R_3)	0.06665	0.02430	0.008727

ERROR OF SURFACE AREA ESTIMATE USING IUR SYSTEMATIC SAMPLING OF DIRECTIONS IN R_3

The surface area can be estimated as twice the mean area of a total projection of the surface (Cauchy formula). Systematic sampling of projection directions usually decreases variance of the estimator. In three dimensional space the three perpendicular directions (ortrip, Mattfeldt et al., 1985, Sandau, 1987), main directions of the cubic lattice used in image analysis (Meyer, 1992, Fig 2), or the sets of normals to Platon solids (Moran, 1944) can be used. Exact upper bounds of variance of surface area estimation using different sets of directions were calculated according to formula in Appendix B.

Fig. 2. Directions of 4-fold, 3-fold and 2-fold axes of cube.

Tab. 2. Coefficient of error (CE) of the Cauchy formula for the estimation of the area of a flat surface in R_3 by averaging areas of projections with equal coefficients:

set of directions	opt.	n	CE	$(CE_1/CE_n)^2/n$
1 direction	*	1	0.57735	1.0
2 orthogonal	*	2	0.30179	1.8
4-fold axes of cube	*	3	0.10163	10.8
3-fold axes of cube	*	4	0.07523	14.7
normals to dodecahedron	*	6	0.03962	35.4
normals to icosahedron		10	0.02444	55.8
2, 3 and 4-fold axes of cube		13	0.01778	81.1

Tab. 3. CE of the Cauchy formula for a flat surface in R_3 using main directions of cubic lattice (axes of cube) with corresponding coefficients obtained by optimisation:

set of axes	opt.	n	CE	$(CE_1/CE_n)^2/n$	4-fold	3-fold	2-fold axes
3 and 4-fold	*	7	0.03374	41.8	0.1374	0.1469	-
2, 3 and 4-fold		13	0.01613	98.6	0.0934	0.0679	0.0746

Tab. 4. CE of the Cauchy formula for a flat surface in R_3 using optimised sets of directions

opt.	n	CE	$(CE_1/CE_n)^2/n$
*	5	0.0604	18
*	8	0.0298	47
*	9	0.0253	58
*	10	0.0211	75
*	13	0.0153	109

The directions with least values of CE for given n in Tabs. 2-4 are marked by asterisks. Values for $n = 2$ and 3 are already known (Sandau, 1987). For $n = 3, 4, 6, 7$ the optimal solutions are highly symmetric. For $n = 5$ the directions form vertices of pentagonal antiprism. For $n = 7$ the directions of normals to the cubooctahedron with two different values of weights were found to be optimal. The optimal solutions with more than 8 directions had dihedral symmetries and unequal weights. The optimisation (Tabs. 3, 4) was performed using the solver in MS Excel.

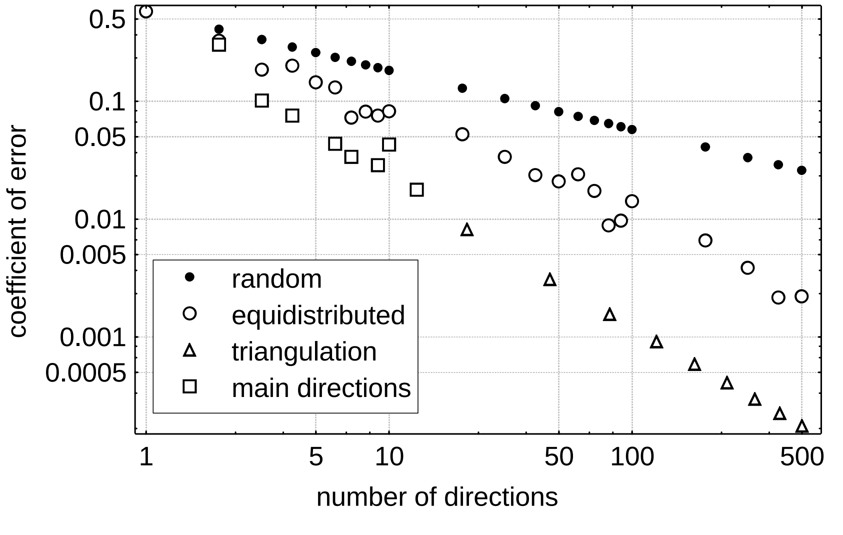

Fig. 3. Coefficient of error of the Cauchy formula for projections of flat surface into systematically chosen directions: mean value of simple random sampling of directions, equidistributed directions, sphere triangulations and main directions of cubic lattice.

The coefficient of error of several roses of directions chosen from two infinite sequences tending to the uniform distribution was evaluated (Fig. 3). The equidistributed sequence was generated by using binary expansion of index of new point to generate its longitude and ternary expansion of index to generate the sine of its latitude (Freulon and

Lantuejoul, 1993). The other sequence was formed by vertices of triangulation of a sphere, obtained by the subdivision of icosahedral triangulation with weights proportional to the sum of surfaces of adjacent triangles (geodesic polyhedra of R. Buckminster Fuller). The main difference between the two methods is that the first generates a regular point pattern in the rectangle and then it maps the pattern to the sphere, which results in partial loss of regularity, while the other generates the regular pattern directly on the sphere.

RESIDUAL ERROR OF SURFACE AREA ESTIMATION USING IUR SPATIAL GRIDS OF LINES

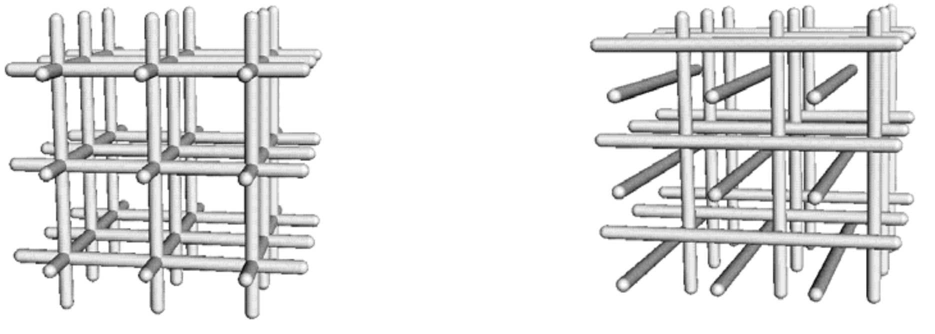

Fig. 4. Orthogonal triplets of fakir probes - spatial grid SG and optimised OTSFP.

Two orthogonal triplets of linear probes were tested: spatial grid by Sandau - SG and halfway shifted orthogonal triple of fakir probes - OTSFP. Counting the intersections of the surface and the spatial grids is equivalent to projecting the surface to orthogonal planes and then counting points of quadratic grids inside the total projections. The trend term in the asymptotic expansion of residual component is proportional to the third power of grid spacing and average perimeter of total projection (Gundersen and Jensen, 1987, Hahn and Sandau, 1989). The mean perimeter of projection of convex body is proportional to the mean width of the body. The trend term of the variance of estimate of a convex body surface area is:

$$\mathrm{var}(\mathrm{est}S(B)) \cong C_G H(B) u^3 \qquad (2)$$

where u is grid spacing and H(B) is the mean width of object under study (Kubínová and Janáček, 1998, Appendix C). The result can be generalized to non-convex objects using total absolute curvature K^3_1 of the object boundary instead of the mean width of the object (Baddeley, 1980), $2\pi H = K^3_1$ for convex objects.

Tab. 5. Values of coefficients in (2) for triple grids of lines (Appendix C).

Grid	C_G
SG	1.8700
OTSFP	0.7332

It can be seen from data in Tab.5 that shifting of linear probes in spatial grid relatively to each other results in about 2.5 times higher efficiency of estimation of area of an isotropic surface.

DISCUSSION

Precision of volume estimators using simple orthogonal grids of either points, lines or planes in various reference spaces can now be compared through the values of the coefficients in the trend term of an asymptotic expansion of the variance of the volume estimators. Exact values of the coefficients are close to results of earlier numerical studies. Results also confirmed that precision of earlier approximations of those coefficients (Cruz-Orive, 1989) based on transitive theory (Matheron, 1965) was sufficient for most practical purposes.

Two sources of variance in surface area estimation using spatial grids were considered similarly as in the paper by Moran (1966) on the precision of a length estimator in the plane, the component due to orientation and the residual component.

Our numerical results on component of variance due to orientation of the grid, the upper bounds for the estimators using various roses of projection directions, complete the earlier results on orthogonal triplet of directions (Sandau, 1987). The efficiency of systematic sampling of directions, calculated for completely anisotropic objects, increases with number of directions: while taking 3 orthogonal directions is better than 30 randomly chosen direction, 13 systematically chosen directions are better than 1000 random directions.

The numerical results concern perfect anisotropy only and are usually too high for real objects with partial anisotropy. Results on partial anisotropy and projections into three orthogonal directions were obtained by simulation studies for ellipsoids (Hahn and Sandau, 1989).

The residual component of variance was calculated for triple orthogonal grids of lines. It is shown that optimization of spatial grid by adjusting position of the linear probes can dramatically influence the residual variance of the surface area estimator. Approximately the same residual variance as with the nonoptimized grid of Sandau (1987) can now be obtained using optimized grid with about 1.4 times greater step.

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APPENDIX A

Measurement of the k-dimensional volume of the grid inside the body K can be expressed as integration of the measure μ. The measure with periods in uZ^d is defined by its Fourier coefficients c_ξ :

$$\text{est } \nu_d(K) = \int_K d\mu(x),\ \mu(x) \approx \sum_{\xi \in u^{-1}Z^d} c_\xi e^{2\pi i(x\xi)}\ , c_{\mathbf{0}} = \mathbf{1}$$

The variance of a grid estimator can be decomposed to the variance due to orientation of the grid (equal to 0 for volume estimators) and the residual variance. According to the Parseval identity the residual variance can be calculated as:

$$\underset{T\in SO_d}{E}\left(\text{var}(\text{est}\,\nu_d(TK)\,|\,T)\right) = \sum_{\xi\in u^{-1}Z^d-\{\mathbf{0}\}} |c_\xi|^2 \underset{T\in SO_d}{E} |\hat{\chi}_{TK}(\xi)|^2$$

where $\hat{\chi}_K(\xi) = \int_K e^{-2\pi i(x\xi)}dx$ is the Fourier transform of the body K and c_ξ are Fourier coefficients of the grid. Let the body K is a sphere B_R with radius R, its Fourier transform can be calculated from Poisson identity:

$$\hat{\chi}_{B_R}(\xi) = (R/|\xi|)^{d/2} J_{d/2}(2\pi R|\xi|)$$

Using the asymptotic expansion of the Bessel function J:

$$J_{d/2}(z) = \sqrt{2/(\pi z)}\left(\cos(z-(d+1)\pi/4) + o(1)\right)$$

we obtain the trend term in the asymptotic expansion of variance for simple spatial grids for u tending to 0:

$$\text{var}(\text{est}\,\nu_d(B_R)) = \sum_{\xi\in u^{-1}Z^{d-k}-\{0\}} c_\xi^2 \cdot \left((R/|\xi|)^{d/2} J_{d/2}(2\pi R|\xi|)\right)^2 \cong \frac{R^{d-1}}{2\pi^2} \sum_{\xi\in Z^{d-k}-\{0\}} \frac{1}{|\xi|^{d+1}} u^{d+1} = C_{d,k}\nu_{d-1}(\delta B_R)u^{d+1}$$

so the coefficient $C_{d,k}$ can be calculated as

$$C_{d,k} = \frac{\Gamma(d/2)}{4\pi^{d/2+2}} Z_{d-k}(d+1)$$

where Z_{d-k} Epstein zeta function corresponding to regular d-k dimensional orthogonal grid. Values of $C_{1,0}$, $C_{2,1}$, $C_{3,2}$, $C_{2,0}$, $C_{3,1}$ and $C_{4,0}$ were calculated using following formulas

$$Z_1(s) = 2\zeta(s) \qquad Z_2(s) = 4L\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}\right) \qquad Z_4(s) = 8\left(1-2^{2-s}\right)\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}-1\right)$$

where Γ is Euler gamma function, ζ is Riemann zeta function and L Dirichlet function (Kendall, 1948). The value of $C_{3,0}$ was calculated by direct summing for small values of points ξ ($|\xi|<100$) and approximating the remainder of the sum by an integral.

APPENDIX B

Let the surface with area S is projected into a rose of directions. Let the distribution function of angles between normals to the surface be $F(\psi)$ and the distribution function of angles between directions of the projection directions be $G(\chi)$. The variance of estimate of the surface from the weighted average of areas of total projections can now be evaluated by double integration of the kernel K_d:

$$K_d(\psi,\chi) = \left(\frac{d\kappa_d}{2\kappa_{d-1}}\right)^2 \underset{T\in SO_d}{E} \left|(Ty,x)(Tu,v)\right| - 1$$

where $\kappa_d = \pi^{d/2}/\Gamma(d/2+1)$ is the volume of the unit ball in R_d, x, y, u, v any unit vectors from R_d such that $\angle(y,u)=\psi$, $\angle(x,v)=\chi$

with respect to the distributions:

$$\mathrm{var}(\mathrm{est}\,S) = S^2 \int\int K_d(\psi,\chi)dF(\psi)dG(\chi)$$

The special case of $\chi=0$ can be easily calculated:

$$K_d(\psi,0) = \left(\frac{d\,\kappa_d}{2\kappa_{d-1}}\right)^2 \frac{2}{d\pi}\left(\sin\psi + \left(\frac{\pi}{2}-\psi\right)\cdot\cos\psi\right) - 1$$

which makes it possible to calculate the coefficient of variance of the Cauchy formula for projection of either a flat hypersurface ($F(\psi)$ is one-point distribution) into the rose of directions or of a hypersurface into a single hyperplane ($G(\chi)$ is one-point distribution). Flat hypersurface has the highest coefficient of variance of estimation of all hypersurfaces.

APPENDIX C

The variance of sphere surface area estimation and its asymptotic expansion using triple orthogonal grids of lines (SG, OTFSP) can be calculated by formulas using Fourier transform of the grids, similar to those in appendix A.

$$\mathrm{var}(\mathrm{est}S(B_R)) = 16 \sum_{\xi\in u^{-1}Z^3-\{0\}} \left|c_\xi\right|^2 \cdot \left(R/|\xi| \cdot J_1(2\pi R|\xi|)\right)^2 \cong \frac{8\cdot R}{\pi^2} \sum_{\xi\in Z^3-\{0\}} \frac{\left|c_\xi\right|^2}{|\xi|^3} u^3$$

Let i, j, k=1..∞, then: $c_{(\pm i,\pm j,\pm k)}=0$, $|c_{(\pm i,\pm j,0)}|^2=|c_{(\pm i,0,\pm k)}|^2=|c_{(0,\pm j,\pm k)}|^2=1/9$

$|c_{(\pm 2i,0,0)}|^2=|c_{(0,\pm 2j,0)}|^2=|c_{(0,0,\pm 2k)}|^2=4/9$ for both SG and OTSFP

$|c_{(\pm(2i-1),0,0)}|^2=|c_{(0,\pm(2j-1),0)}|^2=|c_{(0,0,\pm(2k-1))}|^2=4/9$ for SG and =0 for OTSFP.